Modelos gráficos causales: introducción y algunos de nuestros resultados recientes sobre reglas gráficas sobre ajustes eficientes

Andrea Rotnitzky .

Universidad Di Tella and Harvard T.H. Chan School of Public Health

Based on

Rotnitzky and Smucler, 2020, Journal of Machine Learning Research, 21 188: 1-86,

Smucler, Sapienza and Rotnitzky, 2021, Biometrika, 109, 1, 49–65.

Smucler and Rotnitzky, 2022. *Journal of Causal Inference, 10, 1, 174-189*

Guo, Perkovic and Rotnitzky, 2022, https://arxiv.org/abs/2202.11994

Academia Nacional de Ciencias Economicas, 6 de Septiembre, 2022

Causality in the 21st century

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  - mediation analysis (origin in psychology and sociology),
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  - mediation analysis (origin in psychology and sociology),
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- "Causal revolution" in great part due to the emergence and adoption of two formalisms:
  - Counterfactual Models
  - Graphical Models

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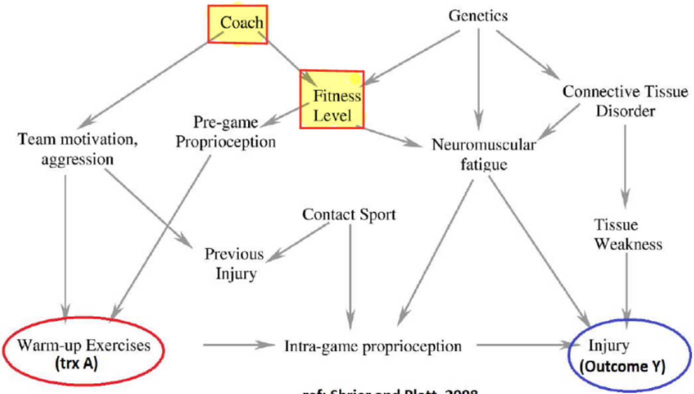
- S*imple graphical rules* exist to explain the potential biases of one's preferred estimation procedure and the possible remedial approaches.
- No graphical rules existed to explain efficiency (variance) in estimation
- In this talk: review graphical models and its use for understanding biases and summarize some of our own work towards filling this gap

An adjustment set



ref: Shrier and Platt, 2008
BMC Medical Research Methodology

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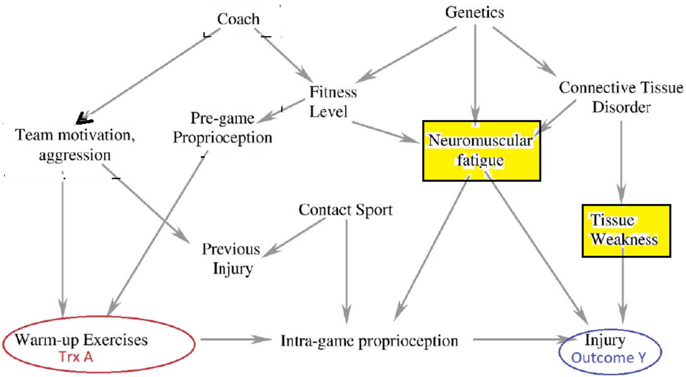

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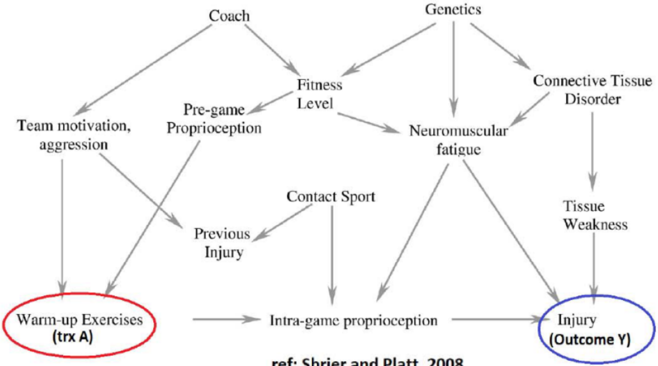
Road map of the talk

- **Gentle introduction to causal graphical models.**
  - Definition and properties
  - Some examples of their use for detecting potential sources of bias
- **Some of our results on efficient adjustment sets**
  - Rules for comparing adjustment sets for point exposure studies
  - Summary of other results
- **Final remarks**

Causal Graphical Models in a nutshell



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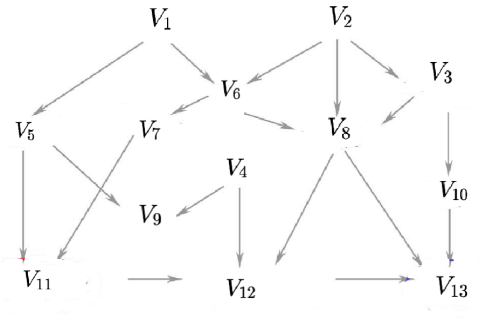

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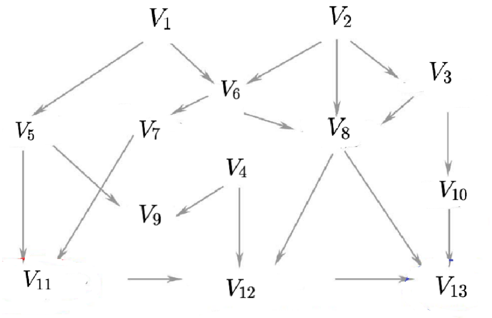

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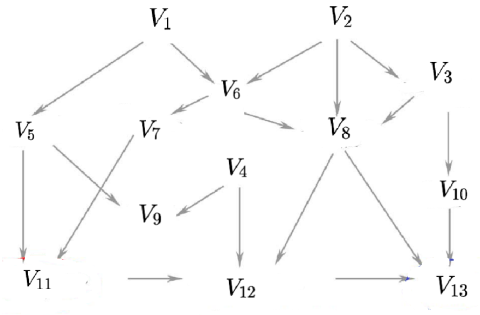

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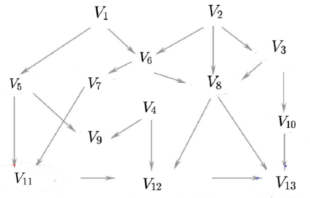

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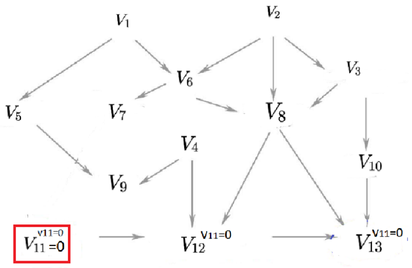
- **Graphical model with independent $\varepsilon_j's$ is tantamount to:**

$$p(\mathbf{v}) = \prod_j p\left(v_j \middle| pa_{\mathcal{G}}(v_j)\right)$$

- The collection of laws for V that factor like this is called a Bayesian Network $\mathcal{B}(\mathcal{G})$.

Causal Graphical Models in a nutshell: counterfactual world static intervention



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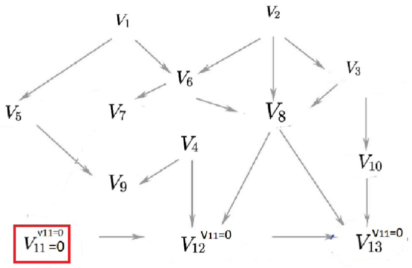

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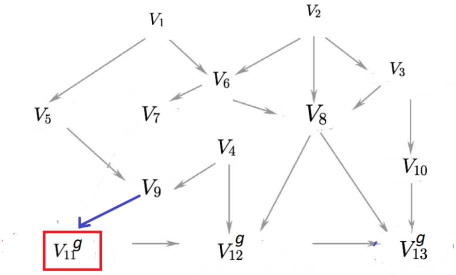
Corollary: counterfactual law is **identified** and given by

$$p_{(v_{11}=0)}(\mathbf{v}) = \prod_{j \neq 11} p\left(v_j | pa_{\mathcal{G}}(v_j)\right) \times I_{\{0\}}(v_{11})$$

Causal Graphical Models in a nutshell: counterfactual world, deterministic dynamic intervention



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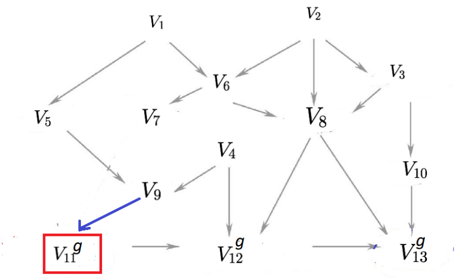

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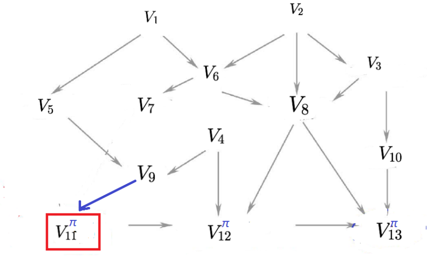

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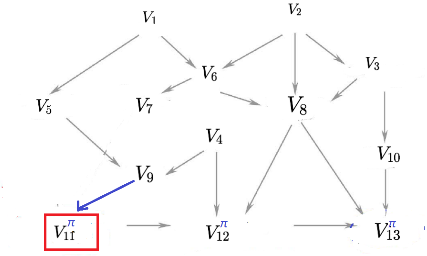

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$$p_{\pi}(\mathbf{v}) = \prod_{j \neq 11} p\left(v_j | pa_{\mathcal{G}}(v_j)\right) \times \pi(v_{11} | v_9)$$

Precursors, review papers in economics and an important reference

- **Pearl's causal graphical model precursors:**
  - **In biology**: Sewall Wright's *linear* structural equations models with normal errors (geneticist) → path analysis
  - **In economics**: Haavelmo's *simultaneous* structural equations model → allows non-recursiveness (simultaneous causation) and assumes parametric equations.
- For a review contrasting Pearl's and Haavelmo's models see Heckman and Pinto (2015). *Causal Analysis After Haavelmo*, Economic Theory.
- See also Imbens (2020) *Potential Outcome and Directed Acyclic Graph Approaches to Causality: Relevance for Empirical Practice in Economics*. Journal of Economic Literature
- For a unifying approach to potential outcomes and causal graphical models see T.S. Richardson, J.M. Robins (2013). *Single World Intervention Graphs (SWIGs): A Unification of the Counterfactual and Graphical Approaches to Causality*, In Foundations and Trends in Machine Learning, ISBN 13: 9781601988102

Causal graphical models

Causal graphical models

a. **Factual world**. The law p of $\mathbf{V} = (V_1, ..., V_J)$ belongs to *Bayesian Network* $\mathcal{B}(\mathcal{G})$, i.e. it factorizes as

$$p(\mathbf{v}) = \prod_{j=1}^{J} p\left(v_j | pa_{\mathcal{G}}(v_j)\right)$$

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b. **Counterfactual world.** For any $\mathbf{A} = (A_1, ..., A_s) \subset \mathbf{V}$, the distrib. of the data when a regime that assigns a_t to A_t with prob. $\pi_t(a_t|\mathbf{Z}_t)$ is implemented in the population (where $\mathbf{Z}_t$ are non-descendants of A_t), is

$$p_\pi(\mathbf{v}) = \prod_{V_j \in \mathbf{V} \setminus \mathbf{A}} p(v_j | pa_{\mathcal{G}}(v_j)) \times \prod_{t=1}^{s} \pi_t(a_t | \mathbf{z}_t)$$

So, p_π is **identified** from p

Bayesian Network

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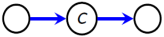
- **Theorem (**Geiger, Verma & Pearl, 1990) **:**

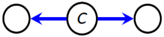
$$A \perp\!\!\!\perp_{\mathcal{G}} B \mid C \Leftrightarrow$$
$$A \text{ is cond. indep. of } B \text{ given } C \text{ under any } p \in \mathcal{B}(\mathcal{G})$$

d-separation

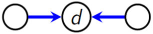
- A, B single vertices, $C \subset V \setminus \{A, B\}$
- a path from A to B is blocked by C if either

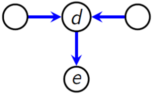
(1) at least one non-collider is in C





(2) $\exists$ at least one collider, such that neither itself nor its descendants is in C





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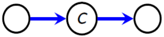

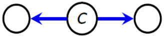

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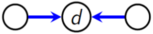

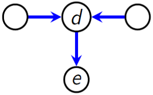

- A and B are d-separated by C if all paths bw A and B are blocked by C

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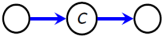

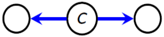

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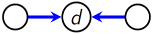

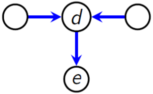

- A and B are d-separated by C if all paths bw A and B are blocked by C
- A set A is d-separated from another set B by $C \subset V \setminus \{A, B\}$ if all $A_j \in A$ and $B_k \in B$ are d-separated by C, in which case we write

$$A \perp\!\!\!\perp_{\mathcal{G}} B \mid C$$

Road map of the talk

- **Gentle introduction to causal graphical models.**
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  - Some examples of their use for detecting potential sources of bias
- **Some of our results on efficient adjustment sets**
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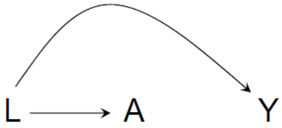
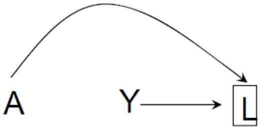
Two potential sources of bias in your causal analysis

NOT conditioning on common causes



Confounding bias

Conditioning on a common effect

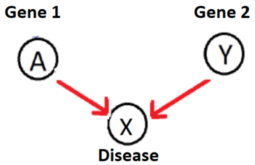


Berkson's bias

Berkson bias

Two variables that are marginally independent will typically be dependent if we condition on a common effect of both variables. (Berkson, 1946)

Example



Suppose $P\left(\text{gene } 1\right) = P\left(\text{gene } 2\right) = 0.02$, genes are marginally independent and Disease if and only if at least one of the two genes is present, i.e.

$$X = 1 - (1 - A)(1 - Y)$$

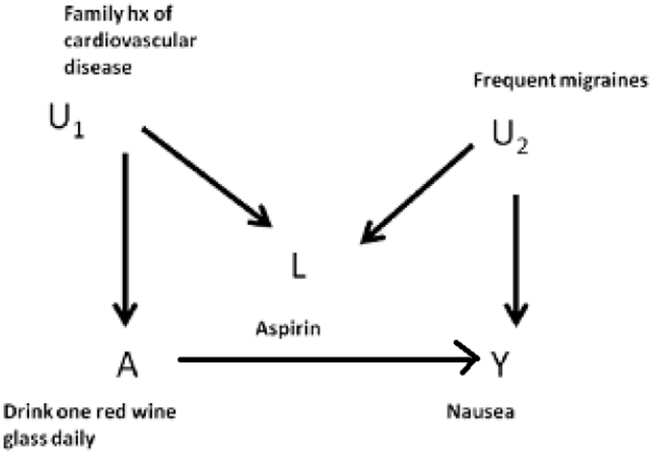
Then,

$$P\left(\text{Gene } 1 | \text{Disease, Not Gene } 2\right) = 1$$

$$P\left(\text{Gene } 1 | \text{Disease, Gene } 2\right) = P\left(\text{Gene } 1 | \text{ Gene } 2\right) = 0.02$$

So, Gene 1 and Gene 2 are negatively correlated conditional on having the disease.

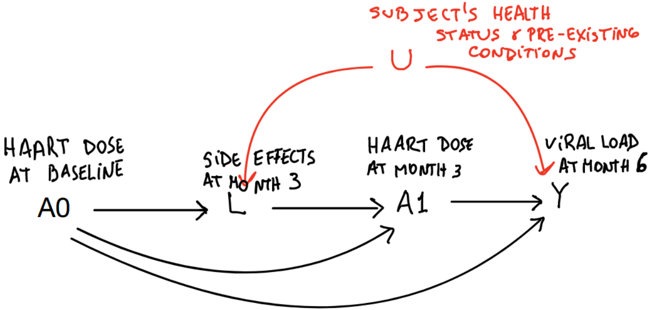
M bias



Family hx of cardiovascular disease
Frequent migraines
U1
U2
L
Aspirin
A
Y
Drink one red wine glass daily
Nausea

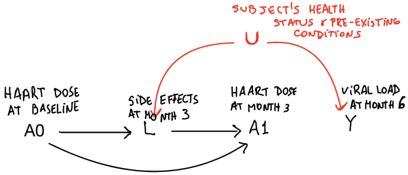
Time dependent confounders

Example: *sequentially randomized trial* of the effect of High vs Low dose of Highly Active Antiretroviral Therapy (HAART) at months 0 and 3 on Viral Load (high vs low) at month 6. (Assume in the graph all variables are binary)



Time dependent confounders

Causal sharp null hypothesis H_0^{causal} that (A_0, A_1) has no causal effect on Y is represented by the graph



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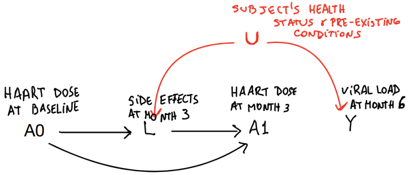

1. **Regression controlling for L fails:** Suppose we fit a saturated (and hence correctly specified) logistic regression model

$$\begin{aligned} \text{logitPr}\,(Y = 1 | A_0, A_1, L) &= A_0\,(\gamma_0 + \gamma_1 L + \gamma_2 A_1 + \gamma_3 A_1 L) \\ &\quad + (\eta_0 + \eta_1 L + \eta_2 A_1 + \eta_3 A_1 L) \end{aligned}$$

and to test H_0^{causal} we test the null hypothesis

$$H_0 : (\gamma_0, \gamma_1, \gamma_2, \gamma_3, \eta_2, \eta_3) = (0, 0, 0, 0, 0, 0)$$

The test does not preserve the $\alpha-$ level because $H_0^{causal} \not\Rightarrow H_0$ since the path $Y - U - L - A_0$ is open when we condition on the collider L.

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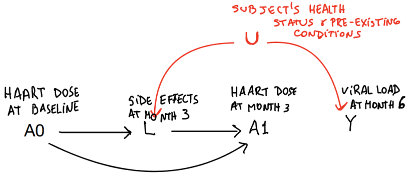

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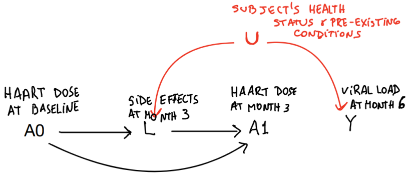

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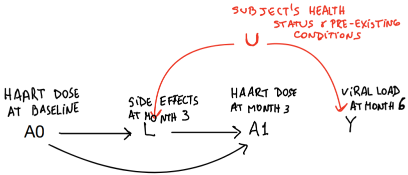


2. **Regression that does not control for L also fails:** Suppose we fit a saturated (and hence correctly specified) logistic regression model

$$\text{logitPr}\left(Y = 1 | A_0, A_1\right) = A_1\left(\alpha_0 + \alpha_1 A_0\right) + \left(\nu_0 + \nu_1 A_0\right)$$

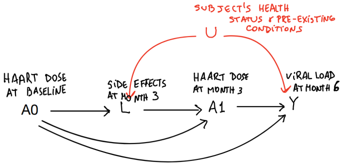
and to test H_0^{causal} we test the null hypothesis

$$H_0^* : (\alpha_0, \alpha_1, \nu_1) = (0, 0, 0)$$

The test does not preserve the $\alpha-$ level because $H_0^{causal} \nRightarrow H_0^*$ since the path $Y - U - L - A_1$ is open when we fail to condition on L.

Identification

- Y_{a_0,a_1} : potential outcome when everybody in the study population takes treatment $A_0 = a_0, A_1 = a_1$.
- **Result** (Robins, 1986): under the causal graphical model represented by the graph



the probability $\Pr(Y_{a_0,a_1} = 1)$ of high viral load in the counterfactual world in which everybody receives treatment $A_0 = a_0, A_1 = a_1$ is identified and given by

$$\Pr(Y_{a_0,a_1} = 1) = \sum_{l=0}^{1} \Pr(Y = 1 | A_0 = a_0, A = a_1, L = l) \Pr(L = l | A_0 = a_0)$$

Road map of the talk

- **Gentle introduction to causal graphical models.**
  - Definition and properties
  - Some examples of their use for detecting potential sources of bias
- **Some of our results on efficient adjustment sets**
  - Rules for comparing adjustment sets for point exposure studies
  - Summary of other results
- **Final remarks**

Counterfactual law under a point exposure intervention

- **Counterfactual law.**

$$p_{\pi}\left(\mathbf{v}\right)=\prod_{j:V_j\in\mathbf{V}\setminus A}p\left(v_j|pa_{\mathcal{G}}\left(v_j\right)\right)\times\pi\left(a|\mathbf{z}\right)$$

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Adjustment formula and adjustment sets

- **Adjustment formula**:

$$\underbrace{E_{\pi}\left[Y\right]}_{\text{intervention mean}} = \underbrace{\sum_{a=0}^{1}\int E\left[Y|A=a,\mathbf{L}=\mathbf{l}\right]\pi\left(a|\mathbf{z}\right)\ p_{\mathbf{L}}\left(\mathbf{l}\right)\ d\mathbf{l}}_{\text{g-functional}}$$

$$= E_{p}\left[\frac{\pi\left(A|\mathbf{Z}\right)}{p\left(A|\mathbf{L}\right)}Y\right]$$

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- **Definition:** A $\mathbf{Z}-$ adjustment set for a single trx A and outcome Y is any $\mathbf{L}$ disjoint with A and Y such that
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- If $\mathbf{Z}=\emptyset$, then we say $\mathbf{L}$ is a *static adjustment set* .

Characterization of Z-adjustment sets

- **Generalized adj. criterion for static (i.e. $\mathbf{Z} = \emptyset$) treatments** (Shpitzer. et. al., 2010, Perkovic et. al., 2015, 2018): $\mathbf{L}$ is static adj. set iff

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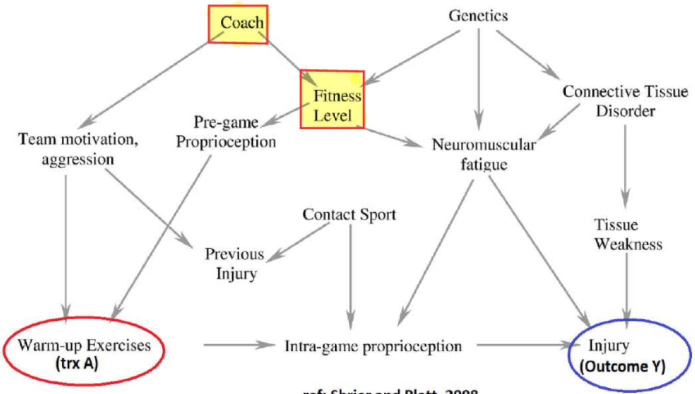
- **Result** (Smucler and Rotnitzky, 2020)**:**

$$\textbf{Class of all } \mathbf{Z} - \textbf{adj sets} = \{\mathbf{L} : \mathbf{L} \text{ is a static adj. set and } \mathbf{Z} \subset \mathbf{L}\}$$

Static adjustment set



ref: Shrier and Platt, 2008
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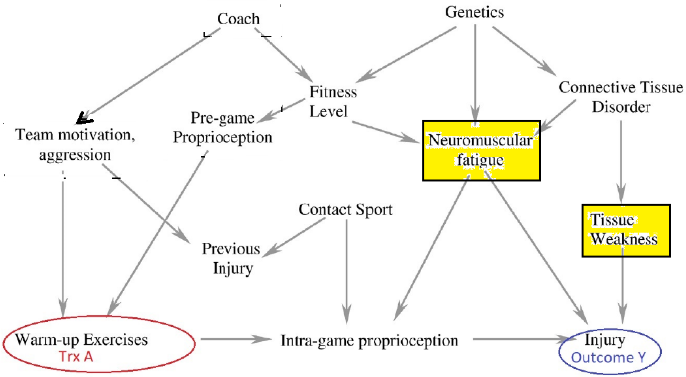
Another static adjustment set



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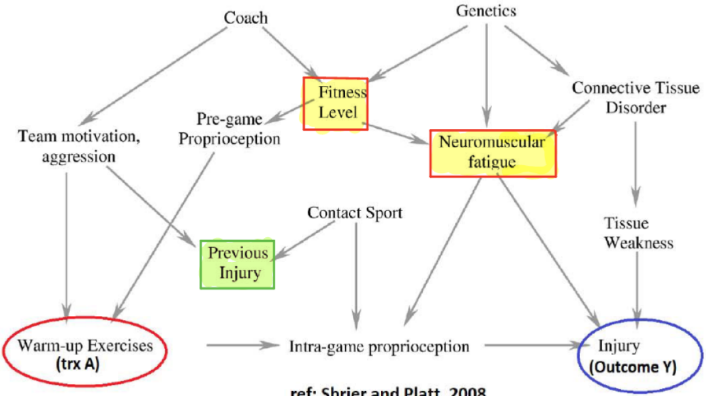
An invalid Z-adjustment , Z= previous injury



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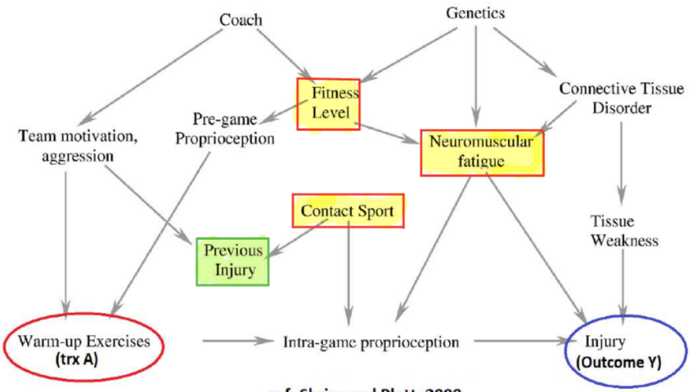

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L-NPA estimators of a counterfactual mean

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$$\psi_{\pi,\mathbf{L}}\left(P\right) \equiv E_{p}\left[\frac{\pi\left(A|\mathbf{Z}\right)}{p\left(A|\mathbf{L}\right)}Y\right] = \text{g-functional that adjusts for } \mathbf{L}$$

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  - Is there a universally optimal adjustment set and, if so, what graphical rules determine it?

Related literature

- Henckel, Perkovic and Maathuis (2019) provided graphical rules
  - for comparing certain pairs of static adjustment sets
  - for determining the globally optimal static adjustment set
- Also, Kuroki and Miyakawa, 2003 and Kuroki and Cai 2004.
- These works assume:
  - causal graphical **linear** model, i.e. $V_j = \beta_j^T pa_{\mathcal{G}}(V_j) + \varepsilon_j, \{\varepsilon_j : j\}$ indep.
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Our contributions

1. **Proved** that Henckel et. al. rules also apply when causal graphical model is agnostic and trx effect estimated via NP $\mathbf{L}-$covariate adjustment .

2. **Derived graphical rules and efficient algorithms for finding**:
   2.1 globally optimal adj. sets for personalized $\mathbf{Z}-$ dependent regimes
   2.2 optimal static and personalized adj. sets among observable adj. sets
   2.3 optimal adj. set subject to a constraint on the sum of the node costs

3. **Extended** rules for comparing adj. sets to time dependent trxs and confounding and proved that optimal time dependent adj. sets do not always exist

4. **Characterized** graphs under which the semip. efficient estimator of the counterfactual mean is asym. equivalent to the optimally adjusted estimator

5. **Derived** an algorithm for identifying the set of all variables in the graph that are informative about the counterfactual mean.

Supplementing adjustment sets with precision variables.

- **Lemma 1.** Suppose $\mathbf{B}$ is a $\mathbf{Z}-$adj. set and $\mathbf{G}$, disjoint with $\mathbf{B}$, satisfies

$$A \perp\!\!\!\perp_{\mathcal{G}} \mathbf{G} \mid \mathbf{B}$$

then, $\mathbf{G} \cup \mathbf{B}$ is *also a* $\mathbf{Z}-$*adj. set* and for all $p \in \mathcal{B}\left(\mathcal{G}\right)$ and all regimes $\pi\left(A|\mathbf{Z}\right)$

$$\sigma^2_{\pi,\mathbf{G}\cup\mathbf{B}}\left(p\right) \leq \sigma^2_{\pi,\mathbf{B}}\left(p\right)$$

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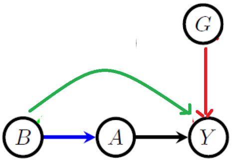
- In particular, for the *static regime* π that sets A to a,

$$\sigma^2_{\pi,\mathbf{B}}(p) - \sigma^2_{\pi,\mathbf{G}\cup\mathbf{B}}(p) = E\left[\left\{\frac{1}{P(A=a|\mathbf{B})} - 1\right\} var\left\{E(Y|A=a,\mathbf{G},\mathbf{B})\middle| A=a,\mathbf{B}\right\}\right]$$



Deleting overadjustment variables

- **Lemma 2.** Suppose $\mathbf{G}\dot{\cup}\mathbf{B}$ is a $\mathbf{Z}-$adj. set and $\mathbf{B}$ satisfies

$$Y \perp\!\!\!\perp_{\mathcal{G}} \mathbf{B} \mid \mathbf{G}, A$$

  If $\mathbf{Z} \subset \mathbf{G}$, then $\mathbf{G}$ is *also a* $\mathbf{Z}-$*adj. set* and for all $p \in \mathcal{B}(\mathcal{G})$ and all regimes $\pi(A|\mathbf{Z})$

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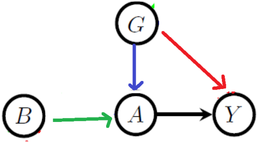
- In particular, for the *static regime* π that sets A to a,

$$\sigma^2_{\pi,\mathbf{G}\cup\mathbf{B}} - \sigma^2_{\pi,\mathbf{G}} = E\left[var(Y|A=a,\mathbf{G})\left\{\frac{1}{P(A=a|\mathbf{B},\mathbf{G})} - \frac{1}{P(A=a|\mathbf{G})}\right\}\right]$$



Comparing two arbitrary adjustment sets

- **Corollary:** Suppose that $\mathbf{G}$ and $\mathbf{B}$ are two $\mathbf{Z}-$adj. sets such that

$$A \perp\!\!\!\perp_{\mathcal{G}} (\mathbf{G} \setminus \mathbf{B}) \mid \mathbf{B}$$

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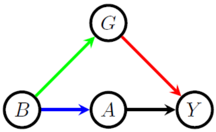
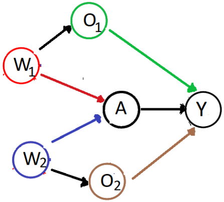
- **Proof:**

$$\sigma^2_{\pi,\mathbf{B}} - \sigma^2_{\pi,\mathbf{G}} = \underbrace{\sigma^2_{\pi,\mathbf{B}} - \sigma^2_{\pi,\mathbf{B}\cup(\mathbf{G}\backslash\mathbf{B})}}_{\substack{\text{gain due to supplementation} \\ \text{with precision component } \mathbf{G}\backslash\mathbf{B}}} + \underbrace{\sigma^2_{\pi,\mathbf{G}\cup(\mathbf{B}\backslash\mathbf{G})} - \sigma^2_{\pi,\mathbf{G}}}_{\substack{\text{gain due to deletion} \\ \text{of noisy component } \mathbf{B}\backslash\mathbf{G}}}$$



Not all adjustment sets are comparable



- (O_1, W_2) is preferable to (O_2, W_1) if green association stronger than brown, and blue association weaker than red
- (O_2, W_1) is preferable to (O_1, W_2) if brown association stronger than green, and red association weaker than blue
- but... (O_1, O_2) is more efficient than both

Optimal adjustment set

- **Theorem:** (Henckel, et. al. (2019)). The set

$$\mathbf{O} = \text{non-descendants of } A \text{ that are parents of } Y \text{ or of vertices in the causal path bw } A \text{ and } Y$$

is a *static* adjustment set. Furthermore, for any other static adjustment set $\mathbf{L}$,

$$A \perp\!\!\!\perp_{\mathcal{G}} (\mathbf{O} \backslash \mathbf{L}) \mid \mathbf{L}$$

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- **Corollary (**Rotnitzky and Smucler, 2020): **O** is the globally optimal static adjustment set.

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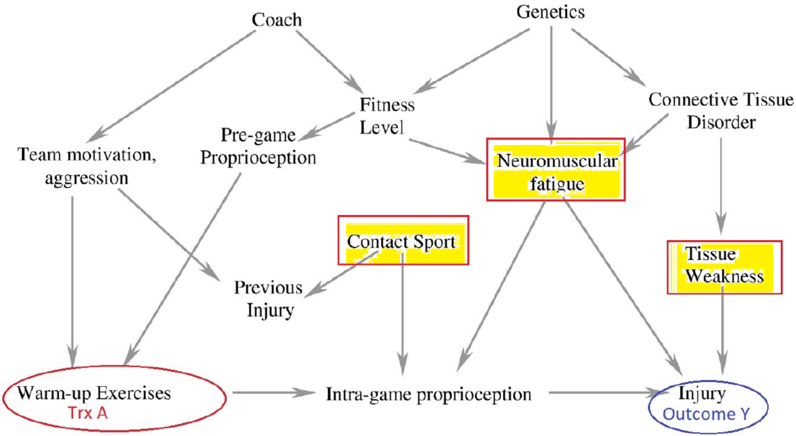
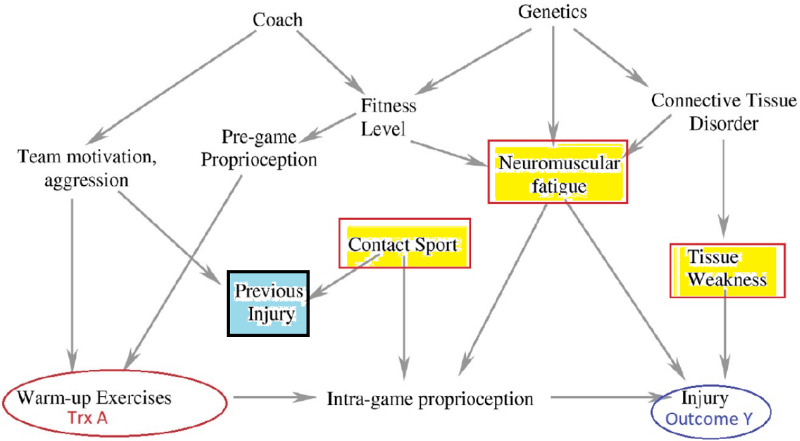
- **Lemma** (Smucler, Sapienza and Rotnitzky, 2021): $\mathbf{O} \cup \mathbf{Z}$ is the globally optimal $\mathbf{Z}$ - adjustment set

Globally optimal static adjustment set



Optimal personalized adjustment set



Road map of the talk

- **Gentle introduction to causal graphical models.**
  - Definition and properties
  - Some examples of their use for detecting potential sources of bias
- **Some of our results on efficient adjustment sets**
  - Rules for comparing adjustment sets for point exposure studies
  - Summary of other results
- **Final remarks**

Our contributions

1. **Proved** that Henckel et. al. rules also apply when causal graphical model is agnostic and trx effect estimated via NP $\mathbf{L}-$covariate adjustment .

2. **Derived graphical rules and efficient algorithms for finding**:
   2.1 globally optimal adj. sets for personalized $\mathbf{Z}-$ dependent regimes
   2.2 optimal static and personalized adj. sets among observable adj. sets
   2.3 optimal adj. set subject to a constraint on the sum of the node costs

3. **Extended** rules for comparing adj. sets to time dependent trxs and confounding and proved that optimal time dependent adj. sets do not always exist

4. **Characterized** graphs under which the semip. efficient estimator of the counterfactual mean is asym. equivalent to the optimally adjusted estimator

5. **Derived** an algorithm for identifying the set of all variables in the graph that are informative about the counterfactual mean.

Graphs with hidden variables

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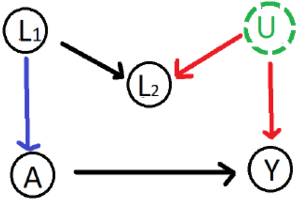

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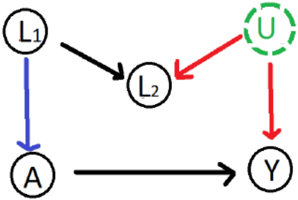

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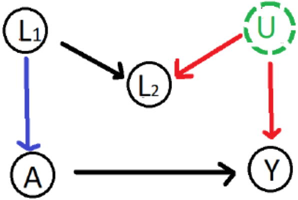

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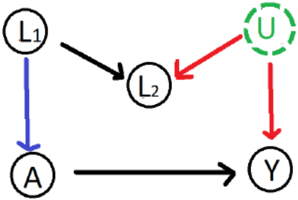

- If U is unobserved, then $\mathbf{L} = \{L_1, L_2\}$ and $\mathbf{L} = \emptyset$ are two valid static adjustment sets which do not dominate each other
  - $\mathbf{L} = \{L_1\}$ is another adj. set but is dominated by $\mathbf{L} = \emptyset$
- In Smucler, Sapienza and Rotnitzky (2021) we characterize sufficient conditions for an optimal **observable** adjustment set to exist

Time dependent treatments

- Suppose A_1 and A_2 are two treatments, $A_1 \in \text{nd}_{\mathcal{G}}(A_2)$. Under a causal graphical model represented by a graph G, the mean of Y_{a_0,a_1} when the static regime that sets A_0 to a_0 and A_1 to a_1 is

$$\begin{aligned} E\left(Y_{a_0,a_1}\right) &= E\left\{\frac{I_{a_0}\left(A_0\right)}{p\left(a_0|pa_{\mathcal{G}}\left(A_0\right)\right)}\frac{I_{a_1}\left(A_1\right)}{p\left(a_1|pa_{\mathcal{G}}\left(A_1\right)\right)}Y\right\} \\ &= E\left\{E\left[E\left[Y|a_0,a_1,pa_{\mathcal{G}}\left(A_0\right),pa_{\mathcal{G}}\left(A_1\right)\right]|a_0,pa_{\mathcal{G}}\left(A_0\right)\right]\right\} \end{aligned}$$

- **Definition:** $\mathbf{L} = (\mathbf{L}_0, \mathbf{L}_1) \subset \mathbf{V}$ is a static time dependent adjustment set relative to trxs (A_0, A_1) and outcome Y in G iff for all $P \in \mathcal{B}(\mathcal{G})$,

$$E\left(Y_{a_0,a_1}\right) = E\left\{E\left[E\left[Y|a_0,a_1,\mathbf{L}_0,\mathbf{L}_1\right]|a_0,\mathbf{L}_0\right]\right\}$$

Time dependent treatments

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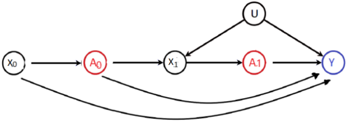
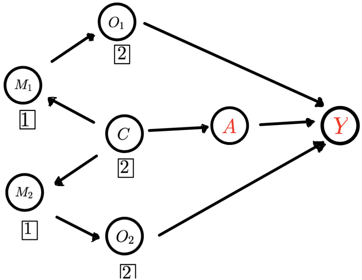


- X_0 is a time 0 adjustment set $(= \mathbf{L}_0)$
- X_1, U and (X_1, U) are time 1 adjustment sets $(= \mathbf{L}_1)$

- In Rotnitzky and Smucler, 2020, we derived rules for comparing static time dependent adjustment sets and showed by example that an optimal adjustment set need not exist.

Study design.

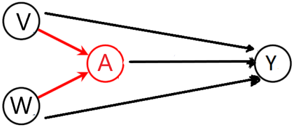
- Assign cost to each graph variable and find the adjustment set leading to smallest estimation variance:
  - subject to a cost constraint → a universal solution does not exist



  - among adjustment sets of minimum cost → for point exposure we provide the universal solution in Smucler and Rotnitzky, 2022, and

    graphical rules for finding it

Semip. efficient estimation vs optimal non-parametric adjusted estimation



- The interventional mean $E\left(Y^{a}\right)$ is

$$E\left[E\left(Y|A=a,V,W\right)\right]=\int E\left(Y|A=a,V=v,W=w\right)\underbrace{p\left(v\right)p\left(w\right)}_{=p\left(v,w\right)}dvdw$$

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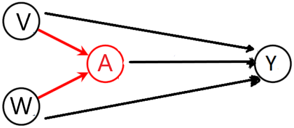

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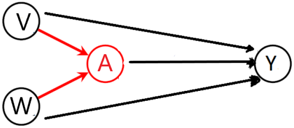

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- Optimal non-parametric adjusted estimator *ignores* restrictions on the marginal law of covariates, i.e. that V and W are marginally independent.

- Semiparametric efficient (SE) *exploits* these restrictions and can be much much more efficient than optimally adjusted NP estimator.

However ... in some graphs the optimally adjusted estimator is efficient

- With discrete data the MLE of $p_a(y)$ under $\mathcal{G}$ is

$$\widehat{p}_{a,MLE}(y) = \sum_{m,o} \mathbb{P}_n(y|m,a)\,\mathbb{P}_n(m|a,o)\,\mathbb{P}_n(o)$$

- Surprisingly, $\widehat{p}_{a,MLE}(y)$ is asym. equivalent to the MLE of $p_a(y)$ under $\mathcal{G}^*$ is

$$\widetilde{p}_{a,MLE}(y) = \sum_{o} \mathbb{P}_n(y|o,a)\,\mathbb{P}_n(o)$$

- In Rotnitzky and Smucler (2000) we characterized the graphs in which the optimally adjusted estimator is semiparametric efficient

Graph reduction for semiparametric efficient estimation of a counterfactual mean

- In Guo, Perkovic and Rotnitzky, 2022, we derived the following.
- Given a graph $\mathcal{G}$ we derived an algorithm that outputs another graph $\mathcal{G}^*$over a subset of the variables in $\mathcal{G}$ such that
  - the semiparametric variance bound for estimation of $E\left(Y_a\right)$ in model $\mathcal{B}\left(\mathcal{G}\right)$ and in model $\mathcal{B}\left(\mathcal{G}^*\right)$ agree
  - $\mathcal{G}^*$ is the smallest such possible graph in the sense that all variables in $\mathcal{G}^*$ are informative. More precisely, the efficient influence function for $E\left(Y_a\right)$ is a function of every variable in $\mathcal{G}^*$ for at least one P in $\mathcal{B}\left(\mathcal{G}^*\right)$

Final remarks

- **Estimation via adjustment vs semip. efficient estimation**:
  - Usual variance/bias trade-off: adjustment relies on less model assumptions
  - Equally or perhaps even more importantly: efficient estimation requires estimation of each cond. density $p\left(V_j|pa_{\mathcal{G}}\left(V_j\right)\right)$. Even debiased, influence-function based, i.e. one-step estimation, will hardly control the estimation bias of these densities.

Open problems

- Inference about the functional returned by the ID algorithm when no observable adj. set exists
  - Some special cases have been studied, e.g. the generalized front door formula, (Fulcher, et. al. 2019). General theory for an arbitrary functional not yet available.
- Optimal adj. sets and efficient estimation for other parameters e.g., trx effect on the treated, and natural direct and indirect effects

THANKS!